

## CHAPTER 2

### BORDER PATTERNS

#### 2.0 Infinity and Repetition

**2.0.1** “What goes (a)round comes (a)round”. You are certainly surprised to see this familiar proverb lying near the beginning, in fact at the very root, of a mathematical book, aren't you? Well, no treatise on fate here, we are simply quoting it literally: if you are moving ‘straight’ on the surface of a sphere or cylinder then you are bound to return to the point where you started from, that's all... This is even more obvious to those who like to think about the structure of our **infinite** universe in terms of space and time, but we are not getting into that, either!

What we have in mind is very earthly indeed: when was the last time you noticed a certain motif **repeating** itself around a vase or belt or the margin (**border**) of a framed photo or ancient mosaic? If you do not quite recall ever having noticed such details, you better be prepared for a change after you go through this book! Such repeating motifs, called **border patterns**, have been with us for a very long time and, rather surprisingly at first, happen to be subject to mathematical rules that are accessible and profound at the same time. We investigate these rules and more with the help of many examples that might even make this book seem like an art book to you: indeed the worlds of art and mathematics are not disjoint!

Before going further, let us point out that infinity and repetition do not always go together. You may recall for example that, while some numbers with an infinite decimal portion have repeating digits after some point (like  $4.7217373\dots = 116,863/24,750$ ), others (like the most famous of all such numbers,  $\pi = 3.141592654\dots$ ) come with a very unpredictable sequence of digits. And, of course, while repeating motifs abound in our finite world, infinite objects exist only in our powerful **imagination**: indeed you will have to train

yourself to see the finite as infinite throughout much of this book; in the case of a border pattern, the easiest way to manage that is to simply wrap it around -- in about the same way that geometers often consider an infinite straight line as a circle of infinite radius!

**2.0.2 Notation.** Although border patterns will be best understood by following the examples and discussion in the following sections, we can briefly state here that they consist of a motif that repeats itself infinitely along a straight line (or finitely along a circle, in view of what we just discussed above). As we will see, there are a total of seven distinct types, each of them equipped with a special four-character '**name**' that always starts with a **p** (for "pattern"). This special notation, even though not terribly important, will be explained as we move through the next seven sections.

## 2.1 Translation left alone (p111)

**2.1.1 Uneventful repetition.** Consider the following pattern, consisting of repeated images of the letter **F**, and try to imagine it either extending itself to the right and to the left for ever or going straight around a 'short' cylinder:



Fig. 2.1

Clearly, a **horizontal translation** by the vector  $\vec{v}$  in figure 2.1 maps the 'first to the left' **F** to the 'second' one, the 'second' one to the 'third' one, and so on; as for that 'first to the left' **F**, you should think of it as being in turn the image of its predecessor (not shown),

etc. Alternatively, we could consider the opposite translation defined by  $-\vec{v}$  that ‘moves’ the **F**s from right to left instead of left to right; or a translation defined by the vector  $3\vec{v}$  that moves every **F** to an **F** three positions to the right, etc. The possibilities for a great variety of translations are endless, and they are all allowed by the letter **F**’s **uniform repetition** along a straight line. But we will usually only consider the ‘minimal’ left-to-right translation defined by  $\vec{v}$ , the pattern’s **minimal translation vector**.

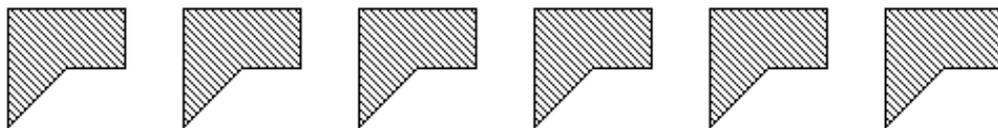
**2.1.2 More than one letters allowed.** Instead of repeating a single letter, as in figure 2.1, we may create patterns by repeating two or more letters or even whole words and more:



Fig. 2.2

Notice that the (minimal) translation vector in figure 2.2 is about twice as long as the translation vector in figure 2.1: the **fundamental region** consists now of “**FAME**” instead of just “**F**”.

**2.1.3 Other motifs.** Instead of repeating letters or words, we may of course repeat any geometrical or other figure of our choice and imagination. Here is an example:



p111

Fig. 2.3

**2.1.4 Attention!** As we will see very shortly, repeating motifs

involves ‘positive risks’: we may end up creating patterns with **more** symmetry (and isometries) than promised by the very title of this section! Remember, this section is devoted to patterns of **p111** type, where **p** stands for translation and the three **1**s denote the **absence** of other isometries to be revealed in the coming sections.

## 2.2 Mirrors galore (pm11)

**2.2.1** Not all letters are created equal. What happens when we try to use the letter **M** instead of **F** in figure 2.1? Let’s see:



Fig. 2.4

Clearly, a vector approximately equal to the vector  $\vec{v}$  of figure 2.1 works as a translation vector for this pattern. But sooner or later one notices something ‘extra’: any vertical line either half way between any two successive **M**s or right through the middle of any **M** acts as a **vertical reflection axis (mirror)** for the entire pattern; that is, the whole pattern remains invariant, with each **M** being reflected onto some other **M**.



Fig. 2.5

Now you are probably ready to protest our claim and argue that only  $m_1$  is a legitimate reflection axis for our pattern, aren’t you?



two lines' or 'between two circles' in figure 2.6 (like  $L_1$  and  $L_2$ , respectively). In more sophisticated terms, mirrors either **bisect** the fundamental region or **separate** two adjacent fundamental regions. Moreover, you may also notice something a bit more subtle: the distance between every two **successive** (hence 'different') mirrors (like  $m_2$  and  $m_3$  in figure 2.5) is equal to **half** the length of the **minimal** translation vector! All these observations are valid in every **pm11** pattern and for fairly deep reasons that will be discussed in chapters 7 and 8, specifically in 7.2.1 and 8.1.5.

**2.2.4 From p111 to pm11.** There is a simple way of turning a **p111** pattern into a **pm11** pattern: simply 'reverse' **every other** motif (as if a mirror ran **through** it)! We illustrate this idea by getting a **pm11** pattern out of the **p111** pattern of figure 2.3:

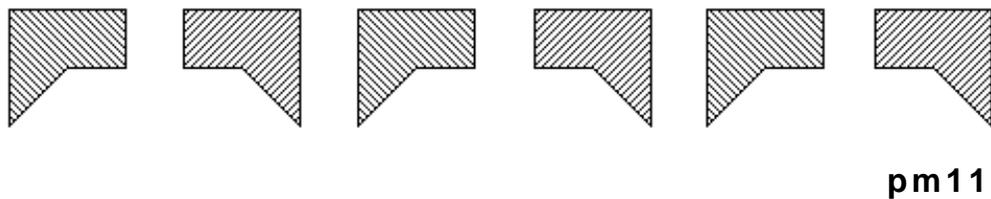


Fig. 2.7

### 2.3 Only one mirror (p1m1)

**2.3.1 An infinite mirror.** Let us now duplicate the letter **D**:

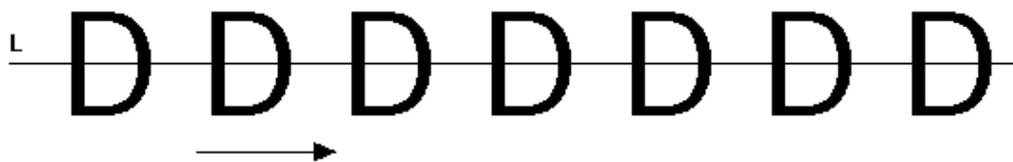


Fig. 2.8

It is obvious that the line **L** that runs through our **D**-pattern acts as a reflection axis for it: indeed the upper half of each **D** is mapped to its lower half (and vice versa), and that happens simply because

the letter **D** itself has **mirror symmetry**. We have just created our third border pattern type, characterized by **horizontal reflection** (and only that, save for the translation of course) and denoted by **p1m1**. (Notice that **m** denotes horizontal reflection when in the third position and vertical reflection when in the second position.)

Here is another example of a **p1m1** pattern using two letters (each of them endowed with a horizontal mirror) instead of one:



Fig. 2.9

**2.3.2 From p111 to p1m1.** It is not necessary to use motifs with horizontal mirror symmetry in order to create a **p1m1** pattern. We may in fact start with an arbitrary **p111** pattern and then reflect it across an axis **parallel** to its '**direction**' to get a perfectly legitimate **p1m1** pattern. Here is how this idea is applied to the pattern from figure 2.3:

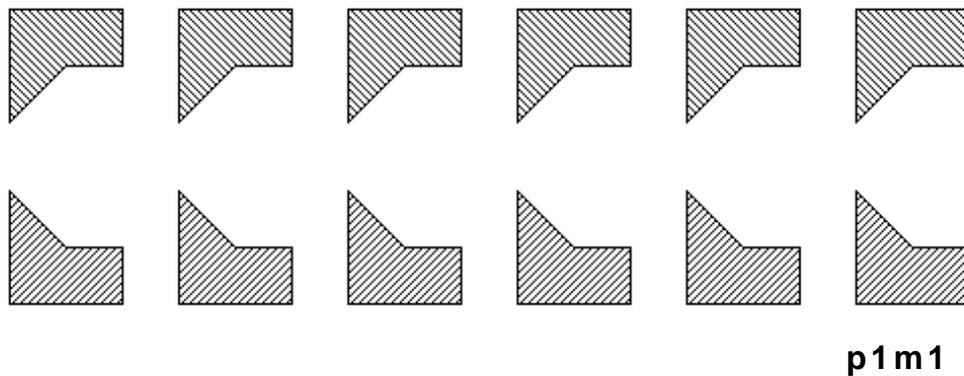


Fig. 2.10

This example simply points to a rather obvious, yet useful, fact: in a **p1m1** border pattern the horizontal reflection axis must be the pattern's '**backbone**' (i.e., the intelligible axis that cuts the pattern into two equal halves, 'top' and 'bottom'); that is, and unlike in the

case of vertical reflection, **there is only one place to look for horizontal reflection!**

**2.3.3 Aesthetic considerations.** We have seen in 2.2.4 and 2.3.2 how simple modifications of the **p111** pattern lead to the **pm11** and **p1m1** patterns. And we have also seen that both the **pm11** and the **p1m1** patterns are created by repetition of a motif that has mirror symmetry: we get a **p1m1** in case the repetition occurs along a direction **parallel** to the motif's internal mirror, and a **pm11** in case the repetition occurs along a direction **perpendicular** to that mirror. This simple geometrical fact bears on the visual impressions created by these patterns: using **arrows** as in figure 2.11, for example, we see that the **p1m1** creates a feeling of **motion** along the pattern's backbone, while the **pm11**'s vertical mirrors create a feeling of **stillness**; as for the **p111** type, it is not unreasonable to say that it stands somewhere between stillness and motion!

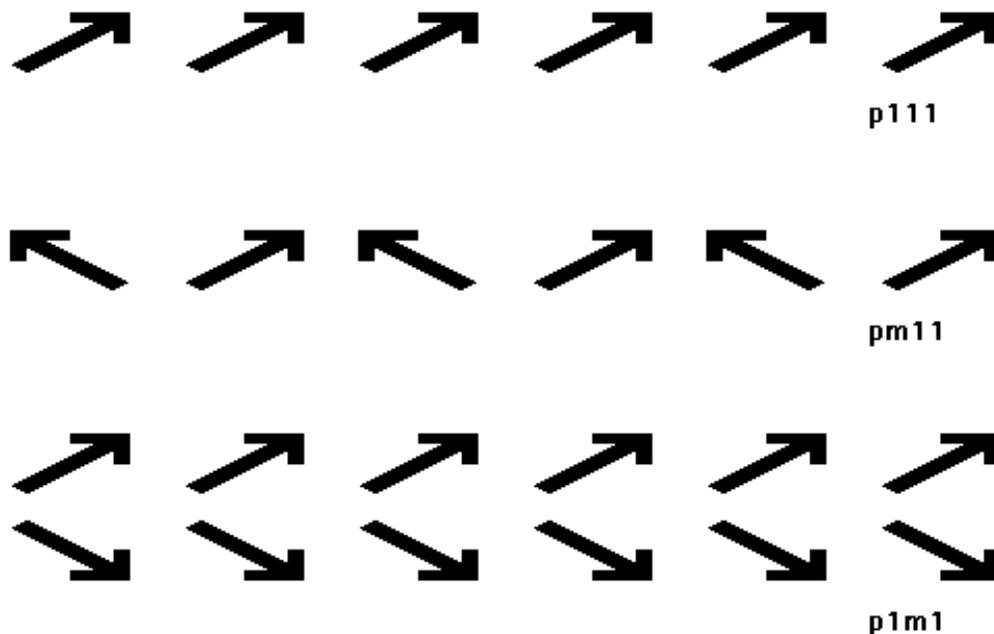


Fig. 2.11

Do you agree with our statements in the preceding paragraph? Well, do not worry in case you do not! When it comes to aesthetics, things are a bit more democratic than in mathematics, and contrasting opinions are allowed to peacefully coexist: simply consider our opinion as a starting point for developing yours! On our

part, we offer a viewpoint that could support either opinion: consider each arrow in figure 2.11 as representing a **footprint** (with the arrow's tip standing for the toes); then the **pm11** pattern can be seen as a series of footprints of people standing on line **next** to each other, while the **p1m1** pattern can be seen as a series of footprints of people standing on line **behind** each other. In fact the **pm11** and **p1m1** patterns may also be created by the footprints of a **jumping** individual, and you can verify this yourself: which way would you move faster, the **pm11** way or the **p1m1** way?

## 2.4 Footsteps (p1a1)

**2.4.1 Moving for sure now!** Consider the arrow-footprint **p1m1** pattern of figure 2.11 'cut in half' as in figure 2.12:



Fig. 2.12

Don't you think that the feeling of motion generated by this pattern is much stronger than the one generated by the 'full pattern' of figure 2.11? With a bit of imagination, you can view the arrows as successive positions of a kayak crossing straight through rough seas! And if you prefer to stay on land, simply return to the **arrow = footprint** equation of 2.3.3 and be proud of yourself: you actually generate that **footstep pattern** many times per day, in fact every time you resort to a straight, steady walk for a few seconds!

**2.4.2 What lies between the footsteps?** Recall that our 'new' pattern has been obtained by 'cutting in half' the **p1m1** pattern of figure 2.11. Moreover, we eliminated precisely those arrows that needed to be eliminated in order to **destroy** horizontal reflection and **preserve** translation at the same time. Notice however that the minimal translation vector (**solid** line) of the new pattern is

precisely **twice as long** as the minimal translation vector (**dotted** line) of the 'old' **p1m1** pattern; this does make sense, as we have indeed eliminated every other arrow:

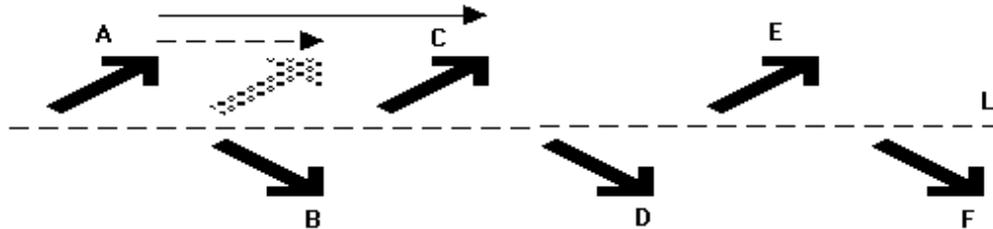


Fig. 2.13

What happens if we **translate** an arrow, say arrow **A** above, by the 'old' vector? Nothing, unless of course we **reflect** it across that between-the-arrows line **L**: then it matches arrow **B**! Repeat the process to arrow **B** -- or first reflect across **L** and then translate by the 'old' vector -- and you get to arrow **C** (which is **A**'s translate by the 'new' vector), and likewise from **C** to **D** (which is **B**'s translate), and so on: our footstep pattern does 'move' thanks to a **glide reflection**! We have just arrived at our fourth border pattern type, characterized by glide reflection and denoted by **p1a1**.

Summarizing our observations, we point out that the glide reflection axis in every **p1a1** pattern (typically denoted by a **dotted** line) runs parallel to the pattern's direction (and **by necessity right through its backbone**, of course); further, the minimal glide reflection vector equals **half** the pattern's minimal translation vector: this reflects on the fact that the glide reflection's '**square**' equals the translation!

**2.4.3 Any good letters out there?** Now that you have understood what a **p1a1** pattern is, can you create one by repeating a **single** English letter, as we did for every border pattern so far? It shouldn't take you that long to realize that this is impossible, even if you resort to letters from distant lands' alphabets or Chinese ideograms! And the reason is simple: while we used letters like **F** (no symmetries), **M** (vertical reflection), and **D** (horizontal reflection) to get the **p111**, **pm11**, and **p1m1** patterns, respectively (in 2.1.1, 2.2.1, and 2.3.1), there is no letter that has glide reflection! More to the point, **no finite figure may ever remain invariant under**

## glide reflection!

Does this mean that there is no way to create a **p1a1** pattern using letters of the English alphabet? Actually not! All we need is **two** English letters mappable to each other by glide reflection:

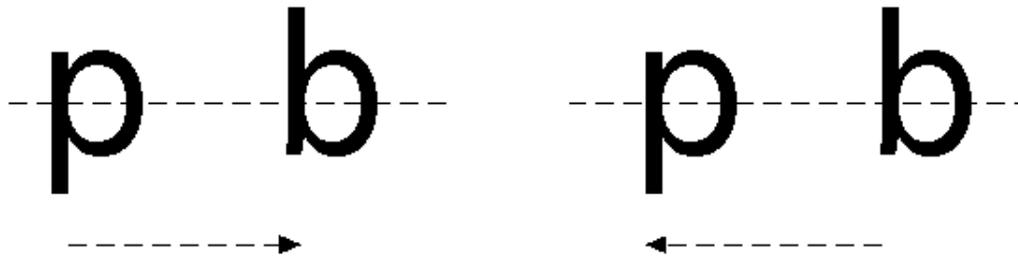


Fig. 2.14

It is not difficult now to create a **p1a1** pattern by infinite repetition of the fundamental region “**p b**”:

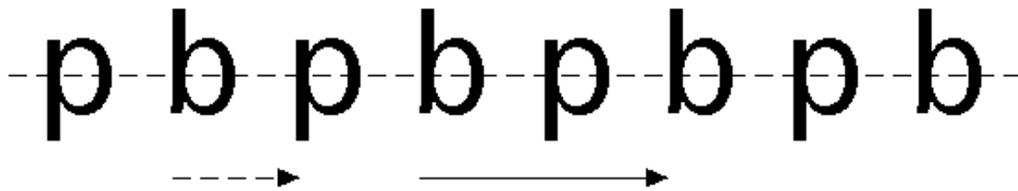


Fig. 2.15

Recall, once again, that all border patterns are **infinite** by definition, but, of course, we can only show a finite part of them on this page, leaving the rest to the imagination. In particular, the rightmost **b** above is mapped by the ‘standard’ left-to-right glide reflection to a **p** right next to it that is not shown, etc.

**2.4.4 Example.** Consider the following ‘arrow pattern’:



Fig. 2.16

What type is it? Does it have glide reflection? It is tempting to say “yes”: arrows **A** and **C** are mapped to arrows **D** and **F** by a ‘**long**’ glide reflection, arrows **B** and **D** are mapped to arrows **C** and **E** by a ‘**short**’ glide reflection, etc. We asked for **one** glide reflection but ended up with **two** instead! Can we still say that there exists glide reflection in our pattern ‘endowed’ with two vectors instead of just one? **No**: a glide reflection is by definition associated with precisely **one vector that works for all motifs** -- otherwise it wouldn’t be an isometry! (Indeed our ‘double vector’ pseudo-glide-reflection above fails, for example, to preserve the distance between the tips of the arrows **A** and **B**, which are ‘mapped’ to the tips of the arrows **D** and **C**, respectively.)

What type is it then? There is clearly some symmetry in our example, in particular a translation mapping **A** to **E**, **B** to **F**, and so on. Could it be just a **p111** then? No, a somewhat closer look shows that there is vertical reflection, with mirrors -- **working for the entire pattern** -- between **A** and **B**, **C** and **D**, **E** and **F**, etc: it’s a **pm11**! (Compare now this **pm11** pattern with the one in 2.3.3: what makes them differ from each other?)

## 2.5 Flipovers (p112)

**2.5.1 One more variation.** Let us revisit the **p1m1** and **p1a1** border patterns in figures 2.6 and 2.15, both of them starting with a **p** and continuing with either a **q** or **b**, respectively. What if we try to continue with a **d** this time? We end up with the following pattern:

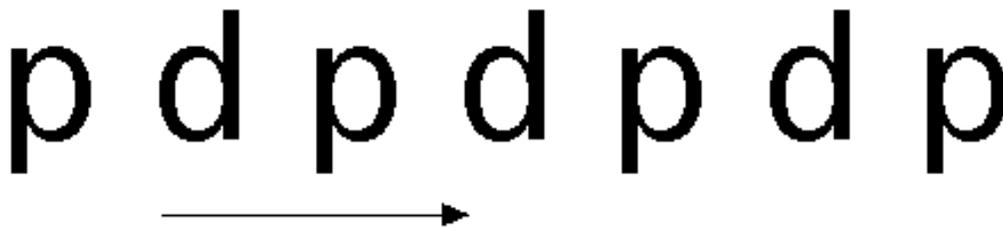


Fig. 2.17

Once again, there seems to be some symmetry involved here, and the pattern is clearly invariant under the indicated translation. You can check that no reflection or glide reflection is going to leave it invariant. There is something else going on though: what happens if you turn this page **upside down**? Does the flipped pattern look any similar to the original one? Have a classmate hold his/her copy **straight** right next to yours in case you cannot remember how the original looked like! And, if that is not possible, just **trace** the pattern and then flip it. What do you think? Is the flipped pattern the same as the original? Well, you may at first say “no”: the original pattern ‘begins’ and ‘ends’ with a **p**, while the flipped one ‘begins’ and ‘ends’ with a **d**... But, do not forget: border patterns are **infinite**, so they do not ‘begin’ or ‘end’ anywhere! With this all-important detail in mind, you must now agree that the original and flipped versions are **identical**!

**2.5.2** How do mathematicians flip? Have you really read 1.3.10 on **half turn** or had you assumed it to be little more than a footnote? Either way we suggest that you quickly review it, so that the special relation between the letters **p** and **d** illustrated in figure 2.18 will make full sense to you:

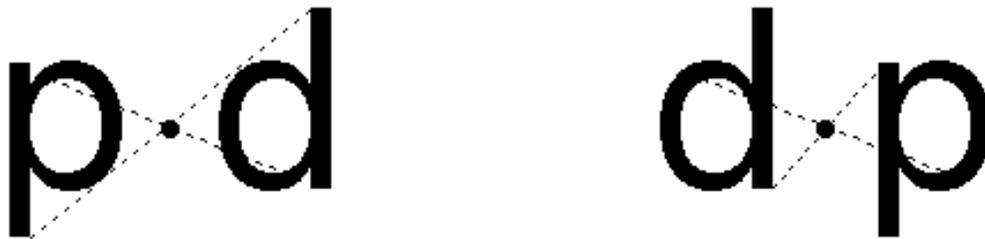


Fig. 2.18

Clearly, **p** and **d** above are images of each other under the shown half turn or **point reflection** (as the **180<sup>0</sup> rotation** was also called in 1.3.10). That is, all we need in order to flip a **p** into **d** or vice versa is a point reflection center, easily found by inspection. It doesn't take that long now to realize that the pattern's **backbone** in figure 2.17 is full of such centers: a half turn around each one of them leaves the **entire** pattern invariant! To confirm this you may like to trace the pattern and then rotate the tracing paper by 180<sup>0</sup>

about your pencil's tip, held firmly at any one of the half turn centers shown in figure 2.19: every **p** on the tracing paper moves on top of a **d** and vice versa!



Fig. 2.19

In particular, our “**p d**” pattern has **half turn** and belongs to the type known as **p112**. Notice that the two **1**s in the second and third positions denote the **lack** of vertical reflection and horizontal reflection (or even glide reflection), respectively; in the same way, the **1** in the fourth position of all types we have seen so far indicated the absence of half turn. As for the **2**, that reflects on the fact that, with  $2 \times 180^\circ = 360^\circ$ , a half turn needs to be applied **twice** -- as its very name aptly suggests -- in order for everything to return to its original position.

**2.5.3 Any single letters?** We now ask the same question we asked in 2.4.3, providing an affirmative answer this time: it is possible to create a **p112** pattern using a single letter. All we have to do is pick a letter that has **internal half turn**, like **N** or **Z**:



Fig. 2.20

Notice that the existence of half turn in the “**Z**” pattern is much more obvious than in the case of the “**p d**” pattern -- why?

**2.5.4 Two kinds of half turn centers.** The **p112** patterns in figures 2.19 and 2.20 have **two kinds** of half turn centers: between either two circles or two lines in the case of the “**p d**” pattern,

right on the center of a **Z** or right between two **Z**s in the case of the “**Z**” pattern. In either case we notice that the distance between any two adjacent (hence of distinct type) half turn centers equals **half** the length of the minimal translation vector. This observation is very much in tune with our remarks in 2.2.3, and we will return to it in 7.5.2.

**2.5.5 Example.** We now return to the pentagon featured in 2.1.3, 2.2.4, and 2.3.2 and show how it may be built into a **p1a1** or **p112**:

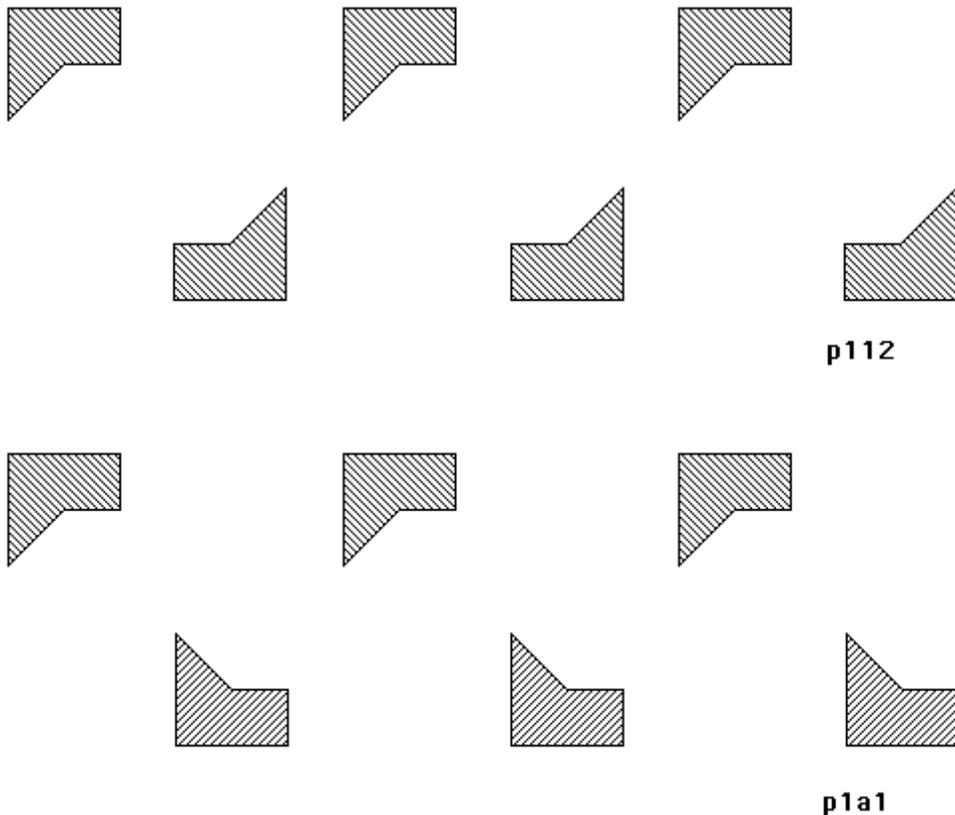


Fig. 2.21

Sometimes students confuse a **p1a1** pattern for a **p112** pattern and vice versa. Comparing the two examples above should help you understand the difference between them even at the ‘intuitive’ level: there is **spinning** (with lots of **parallel segments**) in **p112** as opposed to **straight motion** (and segments going **opposite ways**) in **p1a1**. Also, check what happens to each pattern when you flip it over by rotating the page by 180<sup>0</sup>: in one case (**p112**) the new top row still ‘points’ to the **same** direction (right), while in the other

case (**p1a1**) the new top row 'points' to the **opposite** direction (left). Further, think of what exactly you need to do in each case in order to bring a **tracing paper copy** back to the original pattern!

Anyhow, the best way to distinguish a **p1a1** type from a **p112** type is to remember the isometries that characterize them (glide reflection in **p1a1**, point reflection in **p112**) and be able to explicitly recognize them as such. You may of course wonder: isn't there any way to have **both** these wonderful isometries present in the same pattern? Well, that's the topic of the next section!

## 2.6 Roundtrip footsteps (pma2)

**2.6.1** Are they mutually exclusive? The discussion in 2.5.5 has probably left you with the impression that glide reflection and point reflection cannot quite coexist in a border pattern. In particular, you would probably be ready to guess that the images of any given figure under a glide reflection and under a point reflection must always be distinct. This is not true: those two images could actually be one and the same in some cases! For an example, look at what happens to the letter **V** in figure 2.22:

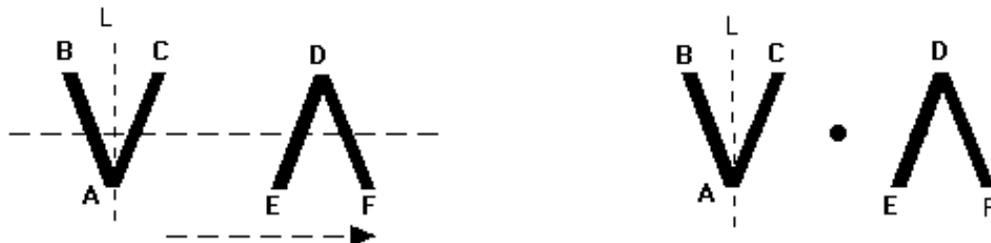


Fig. 2.22

Clearly **V** gets mapped to  $\Lambda$  (capital Greek Lamda) **both** by glide reflection (left) and point reflection (right)! How did that happen? Well, observe that in the case of glide reflection **AB** got mapped to **DE**, and **AC** to **DF**, while in the case of point reflection **AB** and **AC** got mapped to **DF** and **DE**, respectively; notice in the latter case that, consistently with 1.3.10, **DF** and **DE** are parallel to **AB** and **AC**,

respectively. In a way, the two isometries acted on **V** in two very different ways: that should not come as a surprise in view of our remarks in 2.5.5. Were **AB** a bit longer than **AC**, for example, the two images would have been distinct. Likewise, it is important that **AB** and **AC** are not only of equal length, but also at equal distance from the vertical line **L** that bisects **V** and acts as an **internal mirror** for **V**. In short, the effect of the particular point reflection and the particular glide reflection on **V** are **seemingly identical precisely** because **V** has (vertical) **mirror symmetry**!

**2.6.2** All three together now! What happens if we start **repeating** that “**V Λ**” motif created out of **V** in figure 2.22? We end up with the following border pattern:

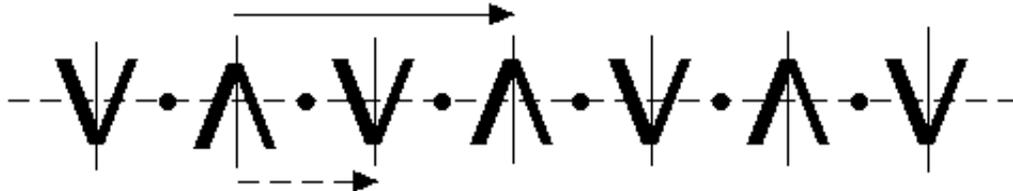


Fig. 2.23

In view of the discussion in 2.6.1, it shouldn't take you long to realize that our “**V Λ**” pattern has vertical reflection (**'inherited'** by individual motifs), glide reflection, and point reflection. Likewise, you should have no difficulty determining the vertical reflection axes, glide reflection vectors, and half turn centers, confirming both figure 2.23 and the remarks made on such entities in 2.2.3, 2.4.2, and 2.5.4. Notice in particular that **half way** between every two adjacent mirrors there exists a half turn center (and vice versa), while the distance between every two adjacent half turn centers (**or** mirrors) is **equal** to the length of the glide reflection vector. Finally, and in view of all the border pattern types and notations you have already seen, you ought to be able to guess this new pattern's 'name': **pma2**.

**2.6.3** Two as good as three! Let us now apply either a half turn or a glide reflection to **pq** and then translate the outcomes

repeatedly, exactly as we did in the previous section; due to the vertical symmetry of **pq**, we end up, in both cases, with the **same pma2** pattern (exactly as it happened with the **V** in 2.6.1 and 2.6.2):



Fig. 2.24

We leave it to you to determine all the isometries of the **pqbd** pattern created in figure 2.24. What is important to observe is that, once again, glide reflection and point reflection seem to ‘**imply**’ each other in the presence of vertical reflection.

What happens if we start with a motif that has point reflection, like **pd**, and then apply either glide reflection or vertical reflection to it, followed by repeated translation? We leave it to you to check that, **either way**, we end up with the **pma2** pattern of figure 2.25:



Fig. 2.25

Again you should determine all the symmetry elements of this **pdbq** pattern and confirm the remarks made in 2.6.2. You also have the right to suspect that, in the presence of point reflection, vertical reflection and glide reflection ‘**imply**’ each other.

What happens when we begin with a **pb** motif, known from 2.4.3 to generate a pattern with glide reflection? Will we still be able to say that, in the presence of glide reflection, point reflection and vertical reflection imply each other? Let’s see... If we apply vertical reflection to **pb** and then we translate, we end up with the following pattern:



**2.6.4 From  $p1a1$  to  $pma2$ .** There is no need for any more  $pma2$  examples, but we would like to justify this section's title! You may recall our 'footstep'  $p1a1$  example in figure 2.12. What happens if that walker returns through exactly the same route? We could very well end up with the following footprint pattern, effectively 'doubling' our  $p1a1$  pattern:

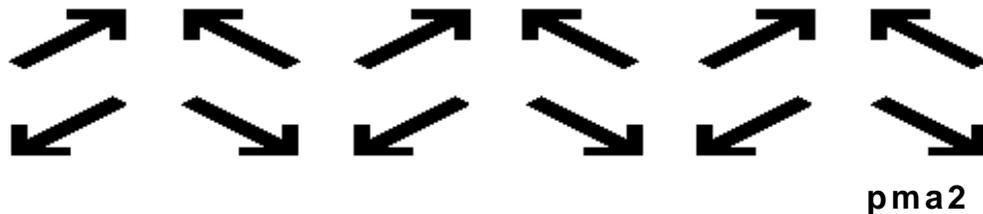


Fig. 2.28

These 'roundtrip footsteps' clearly form a  $pma2$  pattern; glide reflection was known to be there by the pattern's very nature (and discussion in 2.4.1), while the vertical mirrors and half turn centers are even easier to see: just look 'between' the arrows as needed!

**2.6.5 From  $pma2$  to  $p112$  and  $pm11$ .** What happens if we remove every other 'column' of arrows in the pattern of figure 2.28? We simply arrive at a  $p112$  pattern with **all** the half turn centers of the original  $pma2$  pattern preserved (figure 2.29):



Fig. 2.29

Notice also that the **upper half** of the  $pma2$  pattern in figure 2.28 is the familiar  $pm11$  pattern from figure 2.11. That is, every  $pma2$  pattern seems to 'contain' a  $pm11$ , a  $p1a1$ , and a  $p112$ : in view of the isometries involved this observation is not at all

surprising and you should be able to verify it for every **pma2** pattern we have studied. How about the **p1m1** pattern then? Is it ‘contained’ in every **pma2** pattern? Well, as we will see right below, such a possibility is ruled out by the very nature of the patterns involved.

## 2.7 A couple’s roundtrip footsteps (pmm2)

**2.7.1** Is it a ‘new’ pattern? As we pointed out in 1.4.8, every reflection may be viewed as a very special glide reflection the gliding vector of which has length **zero**. What happens to a **pma2** pattern when its glide reflection is ‘upgraded’ to horizontal reflection? Nothing much, in a way; all other isometries will still be there, with the minimum distance between vertical mirrors and half turn centers reduced to **zero**: half turn centers are now found at the **intersections** of the pattern’s **horizontal** reflection axis with every single **vertical** reflection axis! You may confirm all this by looking at a simple example of such a pattern, created by a letter that has **both** vertical and horizontal mirror symmetry:

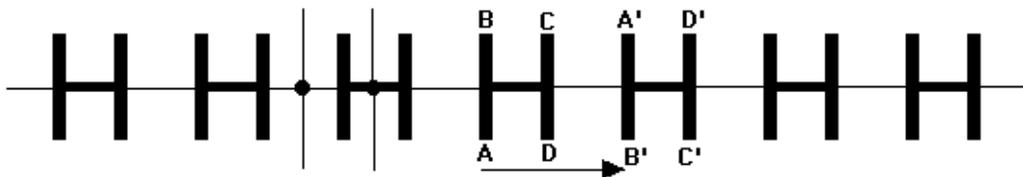


Fig. 2.30

Once again there are **two kinds** of vertical mirrors (right through **Hs** and right between **Hs**), hence two kinds of half turn centers as well. There isn’t really too much new about this pattern, and even its name you should be able to guess: **pmm2**, with first **m** for vertical reflection, second **m** (instead of **a**) for horizontal reflection (instead of glide reflection), and **2** for point reflection.

In addition to viewing the horizontal reflection as a glide reflection with a gliding vector of zero length, we may as well employ it to create glide reflection; this is done by combining the horizontal reflection with the minimal translation vector as shown in figure 2.30: instead of **merely** reflecting each **H** back to itself,

we glide it to the next **H**, too. This idea of using a reflection axis as an axis for a non-trivial, ‘**hidden**’ glide reflection will become very important in future chapters. Notice by the way that the glide reflection of the **p1a1** pattern in figure 2.13 is none other than the ‘hidden’ glide reflection of the **p1m1** pattern in figure 2.11!

**2.7.2 The ‘king’ of border patterns.** The **pmm2** is the ‘richest’ type in terms of symmetry: it ‘contains’ both the **pma2** type (hence, as pointed out in 2.6.5, the **pm11**, **p1a1**, and **p112** types as well) and the **p1m1** type. Indeed we can ‘reduce’ our **pmm2** pattern to either a **pma2** or a **p1m1** pattern by cutting two ‘arms’ off each **H**:

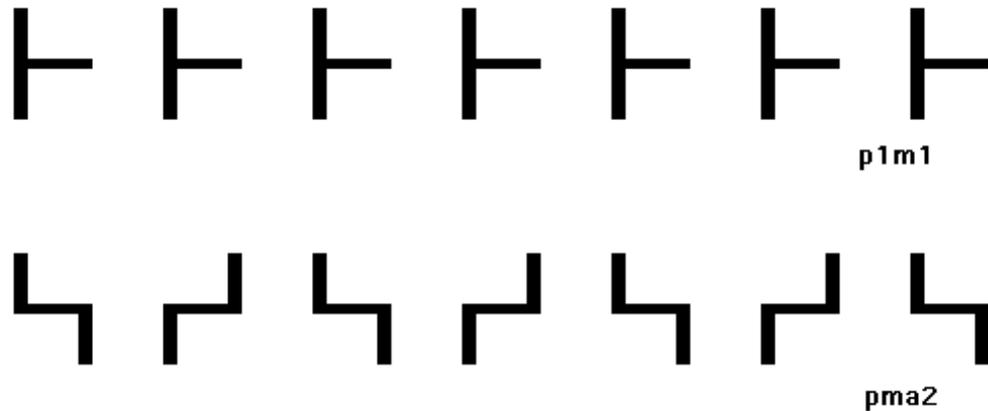
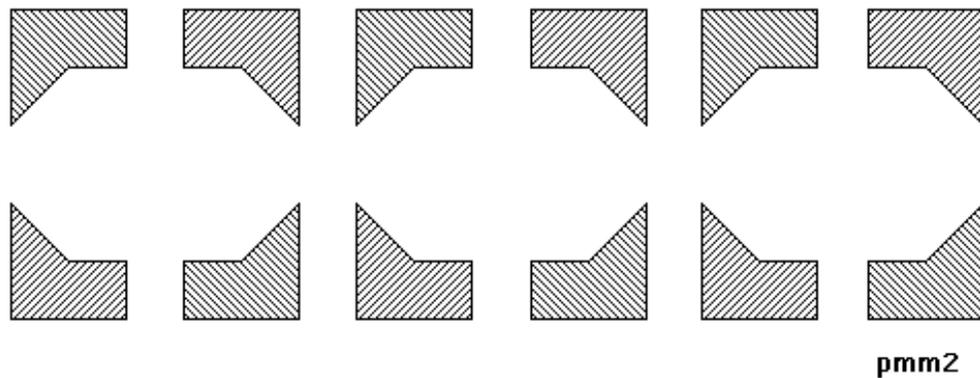


Fig. 2.31

**2.7.3 From pmm2 to pma2.** We now revisit our pentagonal motif and construct **pmm2** and **pma2** patterns as shown below:



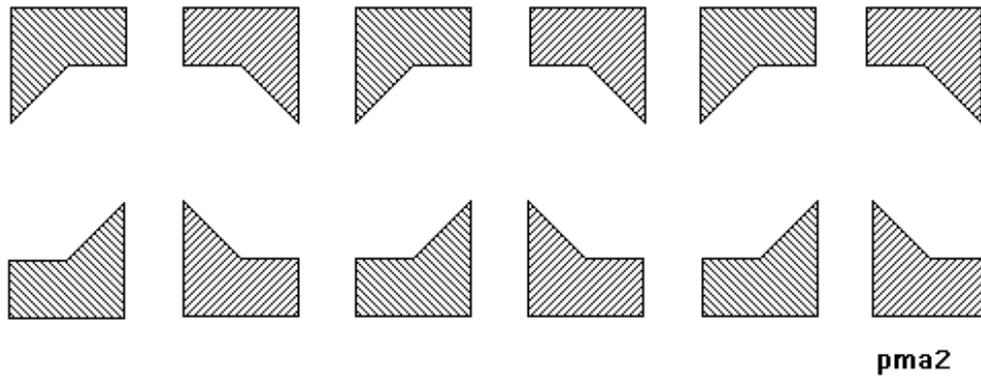


Fig. 2.32

Notice that this time we went from **pmm2** to **pma2** not by cutting the pattern in half (as in figure 2.31) but by **shifting** its bottom row. This 'shifting' will play an important role in chapter 4 and is also at the very root of the fact that the **p1m1** is not 'contained' in the **pma2**.

**2.7.4 More footsteps.** Consider the following 'arrow-footprint' pattern:

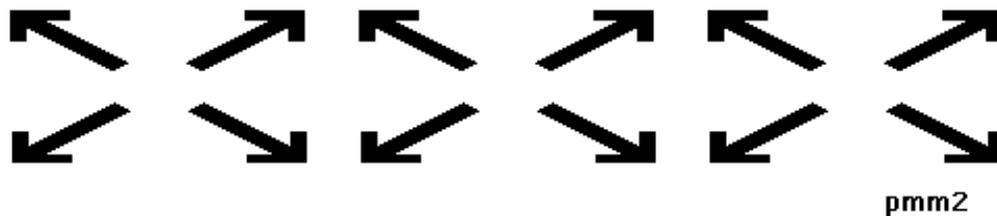


Fig. 2.33

With a little bit of thinking and imagination, you can see this **pmm2** pattern as the **roundtrip** footsteps of a couple walking together -- a bit fast perhaps -- and justify this section's title!

**2.7.5 Footnote.** Our representation of border patterns as footprints and footsteps is partially inspired by a June 24, 1996 **What Shape Are You Into?** lecture delivered at the Art and Mathematics conference at SUNY Albany by eminent Princeton mathematician **John Horton Conway**: he actually demonstrated how to create all seven types 'walking' alone (and barefoot)! You may

like to experiment in that direction, especially when you happen to be alone; can you come up with a footprint representation of the **p112** pattern, alone or not, walking or standing?

Conway has his own **orbifold notation** for border patterns, closely related to his startling **topological** answer to the question discussed right below (and to the harder question of chapter 8, too).

## 2.8 Why only seven types of border patterns?

**2.8.1 Brief summary.** We have so far discussed the following **seven** types of border patterns (with a minimal sequence of English letters generating them (**fundamental region**) in brackets):

**p111**: Translation only (**common to all** seven types) [**F**]

**pm11**: Vertical Reflection [**M**]

**p1m1**: Horizontal Reflection [**D**]

**p1a1**: Glide Reflection [**pb**]

**p112**: Half Turn [**Z**]

**pma2**: Vertical Reflection, Glide Reflection, Half Turn [**pqbd**]

**pmm2**: Vertical Reflection, Horizontal Reflection, Half Turn [**H**]

Are there any other types or ‘combinations’ of border pattern isometries? The answer is “**no**”, and we are in a position to justify this claim without too much extra work.

**2.8.2 Observations.** Based on what we have observed in this chapter, and 2.7.1 & 2.6.3 in particular, we summarize here a number of useful remarks on how a certain isometry or combination of certain isometries implies the existence of another isometry:

[1] Horizontal reflection  $\Rightarrow$  Glide reflection

[2] Glide reflection + Point reflection  $\Rightarrow$  Vertical reflection

[3] Point reflection + Vertical reflection  $\Rightarrow$  Glide reflection

[4] Vertical reflection + Glide reflection  $\Rightarrow$  Point reflection

**2.8.3 Classification.** As we have seen in section 1.5, there exist four types of planar isometries: translation, reflection, rotation, and glide reflection. In the context of border patterns, only isometries that map the border pattern **back to itself** are allowed. That is, translation and glide reflection are allowed only along the pattern's backbone ('horizontally'), reflection may be either horizontal (along the backbone) or vertical (perpendicular to the backbone), and rotation is limited to  $180^0$  (half turn) with its centers lying on the pattern's backbone. Putting ever-present translation aside, we are left with four border pattern isometries, or '**four kinds of reflection**' if you wish: vertical-, horizontal-, glide-, and point-.

Now for every possible border pattern and each one of the four border pattern isometries (and reflection types) discussed above, we may, in fact must, ask a simple question: "does the border pattern have it, or not?" Clearly the answer to each one of the four possible questions is either "yes" or "no". How many possible combinations of answers are there? That will, quite simply, determine an **upper bound** for the number of possible combinations of border pattern isometries and border pattern types: there could be **at most** as many border pattern types as possible combinations of answers!

In theory there are  $2^4 = 16$  possible combinations, precisely because there are **two** possible answers ("yes" or "no") to **four** independent questions -- in the same way that, for example, there exist  $6^4 = 1,296$  possible outcomes when four distinctly colored dice are rolled. In practice, the observations made in 2.8.2 reduce the number of possible combinations to **seven**: that is precisely how many border patterns have been recorded in 2.8.1 and studied in this chapter. In the table below you see the process of elimination, with

**Y** standing for “yes” and **N** for “no” (placed between **question marks** when a negative answer is in fact **impossible** because of one of the observations in 2.8.2); the number inside the parenthesis right next to “impossible” indicates the **applicable observation** from 2.8.2. Whenever a certain combination of answers happens to be impossible for more than one reasons, we cite the ‘simplest’ one.

	vertical	horizontal	glide	point	
	Y	Y	Y	Y	<b>pmm2</b>
	Y	Y	Y	?N?	impossible (4)
	Y	Y	?N?	Y	impossible (1)
	Y	N	Y	Y	<b>pma2</b>
	?N?	Y	Y	Y	impossible (2)
	Y	Y	?N?	N	impossible (1)
	Y	N	Y	?N?	impossible (4)
	Y	N	?N?	Y	impossible (3)
	N	Y	Y	N	<b>p1m1</b>
	N	Y	?N?	Y	impossible (1)
	?N?	N	Y	Y	impossible (2)
	N	N	N	Y	<b>p112</b>
	N	N	Y	N	<b>p1a1</b>
	N	Y	?N?	N	impossible (1)
	Y	N	N	N	<b>pm11</b>
	N	N	N	N	<b>p111</b>

## 2.9 Across borders

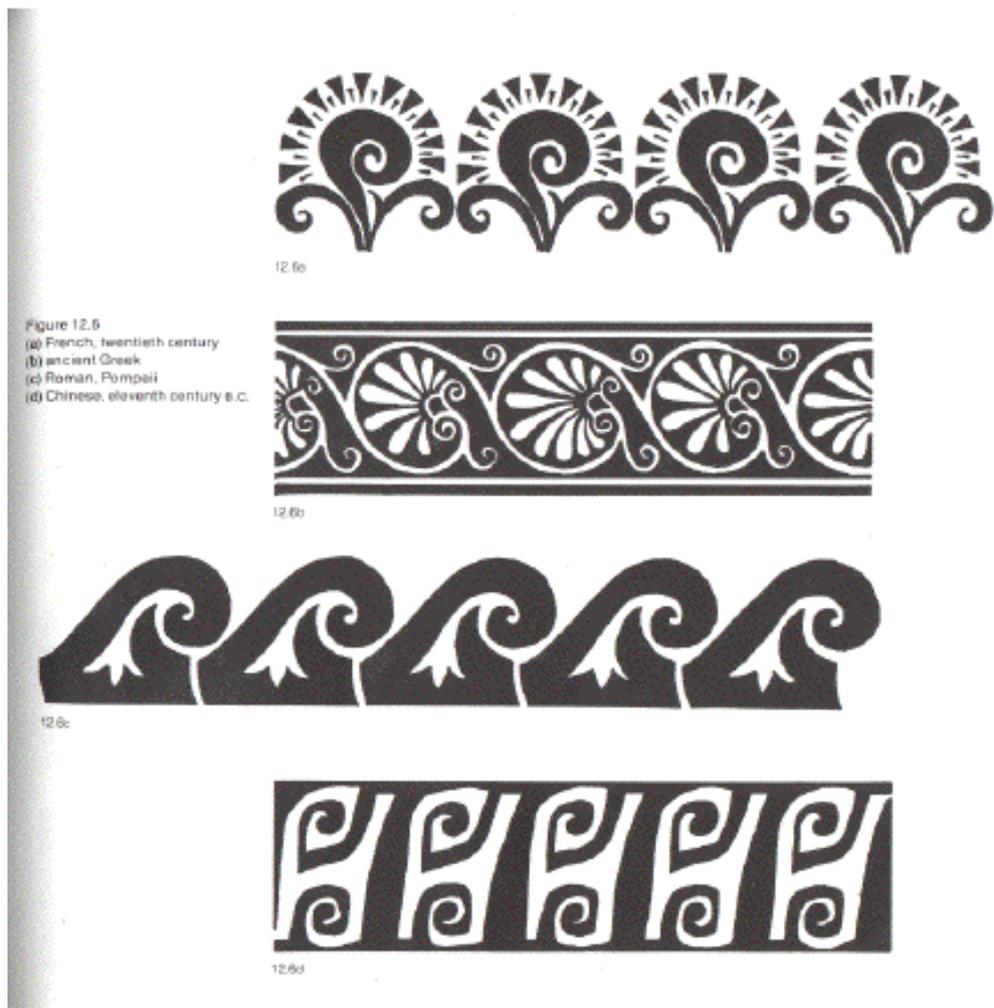
**2.9.1 Mathematics and the artist’s imagination.** Designs that belong to the seven possible types of border patterns are found all over the world, transcending borders, cultures, and historical periods. Two very different looking designs from, say, medieval Europe and pre-Columbian America, designed for very different uses and having very different cultural meanings to their creators, could very well belong to the same type of border pattern. This is not surprising: people, and artists in particular, of varying cultural and technological backgrounds are attracted to symmetry, but symmetry

subjects its unsuspecting worshippers to unspoken mathematical truths and limitations that we just began to explore in this chapter.

Indeed a careful search through art books will reveal the presence of border patterns of any one of the seven types all around the world. You could find the same type around a Roman mosaic or on a Maori wood rafter, for example: different as they may look stylistically, they could very well be the same mathematically. In many cases mathematical kinship is in fact accompanied by stylistic similarity, leading perhaps to conjectures on cultural exchanges between two cultures or periods. While such exchanges and influences definitely existed, stylistic similarities are more likely to be byproducts of the mathematical limitations discussed above.

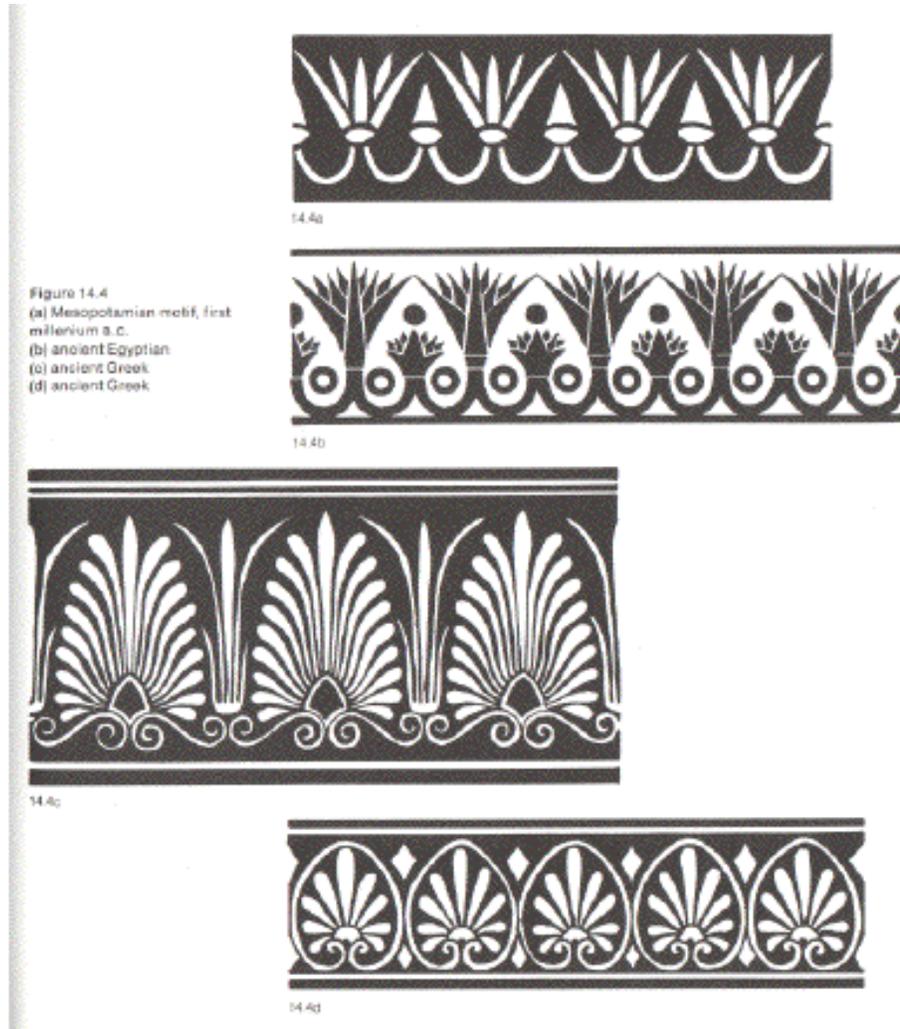
For further discussion on such issues we refer you to the book ***Symmetries of Culture: Theory and Practice of Plane Pattern Analysis***, by Dorothy K. Washburn (an archaeologist) and Donald W. Crowe (a mathematician), published by the University of Washington Press in 1988. The whole book is full of examples of designs from all over the world, while its first chapter discusses both border patterns and wallpaper patterns (which we begin to explore in chapter 4) from the **anthropological perspective**.

Less comprehensive yet brilliantly written and example-oriented is a book written by architect Peter S. Stevens, titled ***Handbook of Regular Patterns: An Introduction to Symmetry in Two Dimensions*** and published by the MIT Press in 1981. Stevens provides several pages of designs from different parts of the world for each border pattern type: going through his book will make you feel that there is nothing but perfectly symmetric designs in our world, which, fortunately or unfortunately, is not quite true. Anyhow, you should from now on be alert and keep an eye open for such 'perfect' designs around you! We give you a jump start here -- and conclude chapter 2 as well -- by citing **seven 'multicultural pages'** from Stevens' book, one for each type of border pattern, and in the same order we studied them; these pages have been included here with official permission from the MIT Press (which also covers a number of figures from Stevens' book included in chapter 4).



p111 border patterns from Peter S. Stevens' *Handbook of Regular Patterns*, figure 12.6, p. 101 (© MIT Press, 1981):

- (12.6a) French, twentieth century
- (12.6b) ancient Greek
- (12.6c) Roman, Pompeii
- (12.6d) Chinese, eleventh century B.C.



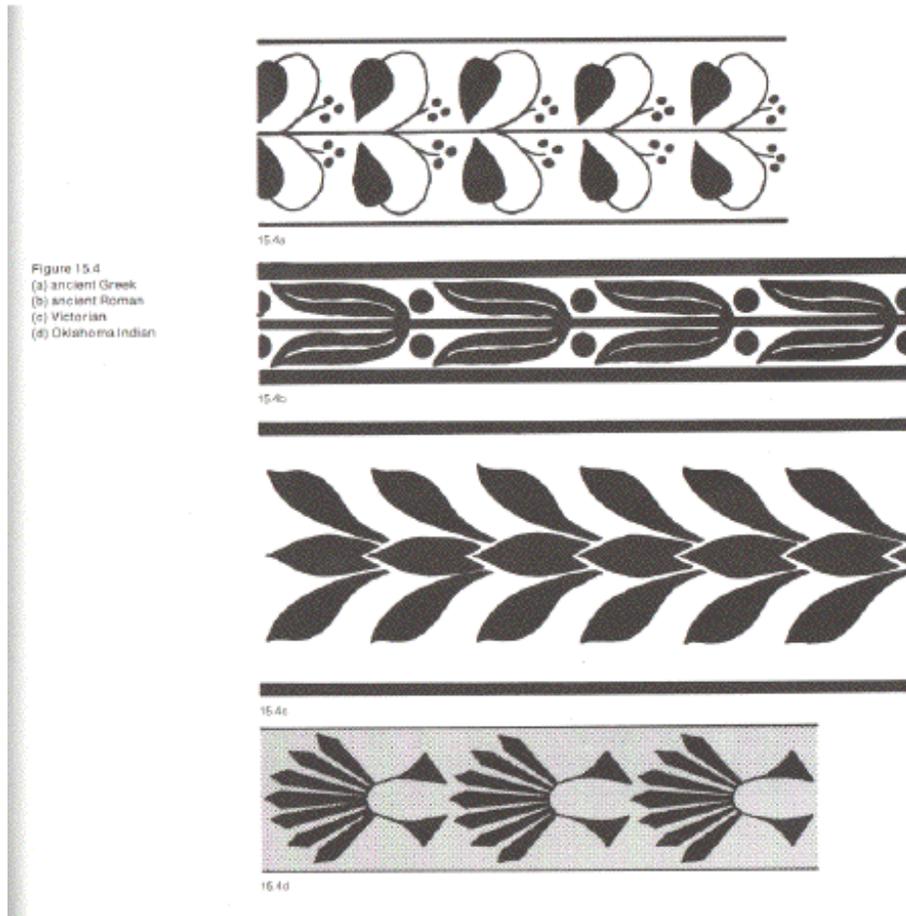
pm11 border patterns from Peter S. Stevens' *Handbook of Regular Patterns*, figure 14.4, p. 121 (© MIT Press, 1981)

(14.4a) Mesopotamian motif, first millennium B.C.

(14.4b) ancient Egyptian

(14.4c) ancient Greek

(14.4d) ancient Greek



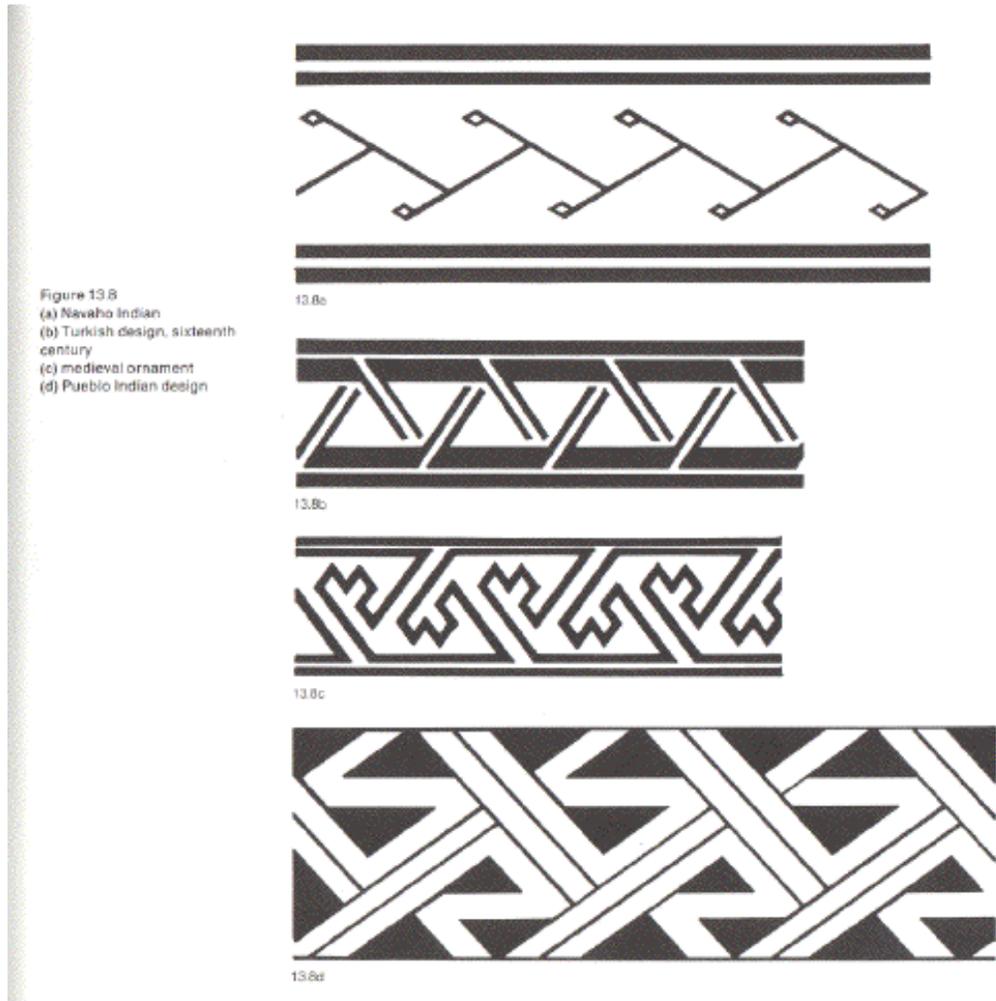
**p1m1** border patterns from Peter S. Stevens' *Handbook of Regular Patterns*, figure 15.4, p. 129 (© MIT Press, 1981)

(15.4a) ancient Greek

(15.4b) ancient Roman

(15.4c) Victorian

(15.4d) Oklahoma Indian



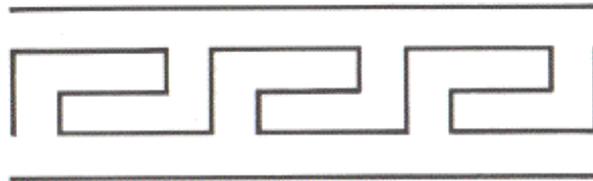
**p1a1** border patterns from Peter S. Stevens' *Handbook of Regular Patterns*, figure 13.8, p. 113 (© MIT Press, 1981)

(13.8a) Navaho Indian

(13.8b) Turkish design, sixteenth century

(13.8c) medieval ornament

(13.8d) Pueblo Indian design



16.7a

Figure 16.7  
(a) border design developed by  
the Chinese, ancient  
Greeks, and Navaho Indians  
(b) ancient Greek  
(c) Turkish  
(d) from pre-Columbian Peru



16.7b



16.7c



16.7d

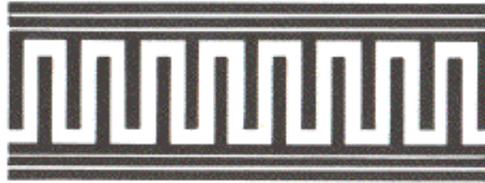
p112 border patterns from Peter S. Stevens' *Handbook of Regular Patterns*, figure 16.7, p. 142 (© MIT Press, 1981)

(16.7a) border design developed by the Chinese, ancient Greeks, and Navaho Indians

(16.7b) ancient Greek

(16.7c) Turkish

(16.7d) from pre-Columbian Peru



17.5a

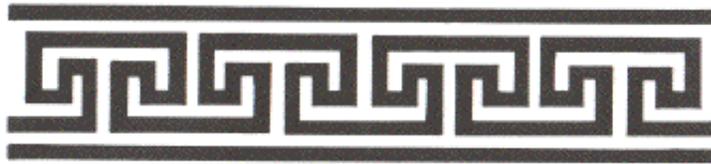
Figure 17.5  
(a) ancient Greek  
(b) French, Louis XV  
(c) Chinese, as well as ancient  
Greek  
(d), (e) Chinese



17.5b



17.5c



17.5d



17.5e

**pma2** border patterns from Peter S. Stevens' *Handbook of Regular Patterns*, figure 17.5, p. 152 (© MIT Press, 1981)

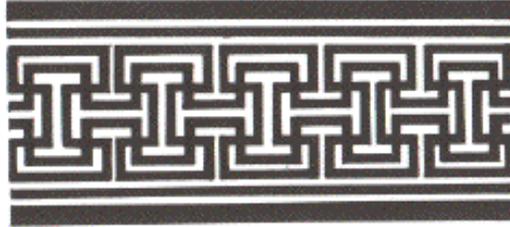
(17.5a) ancient Greek

(17.5b) French, Louis XV

(17.5c) Chinese, as well as ancient Greek

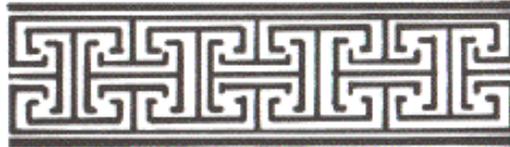
(17.5d) Chinese

(17.5e) Chinese

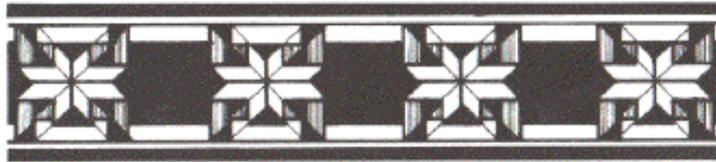


18.6a

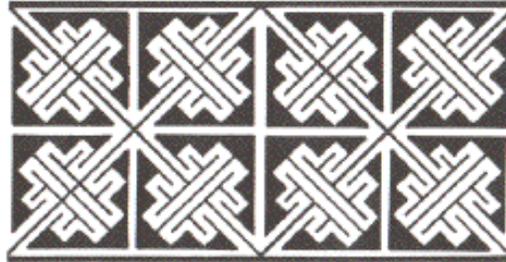
Figure 18.6  
(a) Pompeian mosaic  
(b) medieval  
(c) medieval  
(d) Celtic manuscript design



18.6b



18.6c



18.6d

**pmm2** border patterns from Peter S. Stevens' *Handbook of Regular Patterns*, figure 18.6, p. 162 (© MIT Press, 1981)

(18.6a) Pompeian mosaic

(18.6b) medieval

(18.6c) medieval

(18.6d) Celtic manuscript design